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SHORT REVIEW

Ratio Relationships between π, the Fine Structure Constant and the Frequency Equivalents of an Electron, the Bohr Radius, the Ionization Energy of Hydrogen, and the Classical Electron Radius

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Abstract: This paper demonstrates and analyzes the multiple and complicated ratio inter-relationships among pi, π; the fine structure constant, α; the frequency equivalents of an electron, v_e; the Bohr radius, v_{a0}; the ionization energy of hydrogen or the Rydberg constant, R; v_R; and the classical electron radius, v_{re}. These relationships are partially known, but many are not yet fully recognized. This paper presents all of them as a unified system, demonstrating an unexpected inter-relationship among the quantum properties of hydrogen that can be used to calculate and understand relationships between fundamental constants. These ratios are of logical origin, but are quite complicated and not intuitive. These relationships stem from geometry and the nature of the components of hydrogen: for example, they relate a transition of a mass/energy to a circumference distance. α is related to six different frequency equivalent ratios: the product of 4π and v_R, divided by v_{a0}; the square root of 2 times v_R divided by v_e; v_{a0} divided by the product of 2π times v_e; the product of 2π and v_e divided by v_{re}; the square root of the ratio of v_{a0} divided by v_{re}; and the cube root of 4π times v_R divided by v_{re}. π is also related to multiple ratios of α and the hydrogen frequency equivalents from the previous relationships. $8\pi^2$ is equal to the ratio of v_{a0} squared divided by the product of v_R and v_e. The combination products of α, 2, and 2π are the scaling ratios of the transformations between a mass/energy and a distance for all of the four hydrogen constants. If any three of the six constants are known, then the other three can be derived, demonstrating the intricate relationship of these constants. These relationships also must hold for effective α (running α) as well. In part, they explain its origin and why α must change in certain settings. Therefore these relationships extend to high energy physics, as well as adding new insights to effective α.

Keywords: fine structure constant, effective fine structure constant, electron, Bohr radius, Rydberg constant, fundamental constants, pi, coupling constants

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Introduction

It is common to encounter inter-relationships between fundamental constants and π. Another physical ratio dimensionless constant, the fine structure constant, α,[1–5] is very commonly encountered in many different physical settings. This paper explores the ratio inter- relationships of these two constants and the quantum properties of hydrogen. Since α and π are both dimensionless number ratios, all the evaluated properties of hydrogen are converted to frequency equivalents and evaluated as dimensionless ratios.[6–8] The entities are evaluated as frequency equivalent ratios and include an electron, v_e; the Bohr radius, v_{a0}; and the ionization energy of hydrogen or the Rydberg constant, R; v_R; and the classical electron radius, v_{re}.

It will be shown that these frequency equivalents related to the components defining hydrogen are directly related to π and α. Some of these relationships represent manifestations of classic Euclidean geometric, but they are within a quantum system across physical constant units of energy/mass and distance. Though complicated and not intuitive, these relationships are made more readily apparent by using one unit, Hz. The relationships are important because they demonstrate the inherent fundamental inter- connections between the properties of hydrogen and many other entities.[9,10] The relationships must also be fulfilled for effective α, or running α, so they extend beyond hydrogen into high energy physics as well.[1–3] These findings offer new insights into the nature and origin of effective α.

Methods and Results

All of the fundamental constants are converted to frequency equivalents.[6–8] The masses are converted by multiplying by c^2 (speed of light squared) then dividing by h (the Planck constant). The distances are converted by dividing the wavelength into c. Energies are converted by dividing by h. v_e equals 1.2355899×10^{20} Hz, v_{a0} equals 5.6652564×10^{18} Hz, and v_R equals $3.28984196 \times 10^{15}$ Hz. The classical electron radius, v_{re} equals 1.0638709×10^{23} Hz. All of the data for the fundamental constants are taken from http://physics.nist.gov/cuu/Constants/.

Equation (1) is the classic Coulomb's law, where F equals the force, k is Coulomb's constant, and e^2 is the square of the unit charge of an electron, divided by the distance separating the charges squared, λ^2. In this paper the binding energy of the electron in hydrogen is used as the unified quantum phenomenon. Equation (2) evaluates the energy of a system rather than the force. A different representation of Coulomb's constant, k, is utilized with components of the speed of light squared, c^2, and the magnetic constant, μ_0. In Equation (3) both sides of Equation (2) are divided by Planck's constant, h, which converts the energies into frequency equivalents. In this case the frequency equivalent of the electron binding energy of hydrogen, v_R, is shown. v_R is related to the Rydberg constant, R. Rearranging the other components converts the Bohr radius, λ_{a0}, to v_{a0}, as seen in Equation (4).

$$F = \frac{ke^2}{\lambda^2} \tag{1}$$

$$E_R = \frac{c^2\mu_0}{4\pi}\frac{e^2}{2\lambda_{a0}} \tag{2}$$

$$\frac{E_R}{h} = \frac{c\mu_0 e^2}{h}\frac{1}{8\pi}\frac{c}{\lambda_{a0}} \tag{3}$$

$$v_R = \frac{c\mu_0 e^2}{h}\frac{1}{8\pi}v_{a0} \tag{4}$$

Equation (5) demonstrates that the dimensionless product of c, μ_0, and e^2, divided by h is equal to 2α.

$$2\alpha = \frac{c\mu_0 e^2}{h} \tag{5}$$

Substituting back into equation (4) leads to equation (6), where α is related to 4π times v_R divided by v_{a0}. Equation (7) equates π to v_{a0} times α divided by 4 times v_R.

$$\alpha = \frac{4\pi v_R}{v_{a0}} \tag{6}$$

$$\pi = \frac{v_{a0}\alpha}{4v_R} \tag{7}$$

It is known that the ionization energy of hydrogen is conceptually related to the change in the velocity of an electron from the annihilation velocity of c to velocity αc, Equation (8). Dividing the energies on both sides of the equation by h converts this relationship to frequency equivalents, Equation (9). Equations (10 and 11) solve for α. There is no π in this Equation. α^2 divided by 2 equals the ratio of ν_R divided by ν_e. Rearranging equation (10) solves for ν_R, Equation (12). Substituting this into equation (7) demonstrates that α equals ν_{a0} divided by the product of 2π and ν_e, Equation (13). π is solved for in Equation (14), ν_{a0} divided by 2 times α times ν_e.

$$E_R = \frac{m_e c^2 \alpha^2}{2} \tag{8}$$

$$\frac{E_R}{h} = \nu_R = \frac{m_e c^2}{h}\frac{\alpha^2}{2} = \nu_e \frac{\alpha^2}{2} \tag{9}$$

$$\frac{\alpha^2}{2} = \frac{\nu_R}{\nu_e} = \frac{\nu_{a0}}{\nu_e}\frac{\nu_R}{\nu_{a0}} \tag{10}$$

$$\alpha = \sqrt{\frac{2\nu_R}{\nu_e}} \tag{11}$$

$$\nu_R = \frac{\nu_e \alpha^2}{2} \tag{12}$$

$$\alpha = \frac{\nu_{a0}}{2\pi\nu_e} \tag{13}$$

$$\pi = \frac{\nu_{a0}}{2\alpha\nu_e} \tag{14}$$

The ratio of ν_{a0} divided by ν_e, divided by the ratio of ν_R divided by ν_{a0}, cancels out the α factors, leaving only $8\pi^2$, Equation (15). This is equal to ν_{a0} squared divided by the product of ν_e and ν_R, Equation (15). Equation (16) solves for π solely from quantum constants of hydrogen and the integer 2.

$$8\pi^2 = \frac{\nu_{a0}^{\;2}}{\nu_e \nu_R} = \frac{\left(\frac{\nu_{a0}}{\nu_e}\right)}{\left(\frac{\nu_R}{\nu_{a0}}\right)} \tag{15}$$

$$\pi = \frac{\nu_{a0}}{2\sqrt{2\nu_e \nu_R}} \tag{16}$$

r_e is related to the distance separating two unit charges with a potential annihilation energy equal to the mass of an electron, Equation (17). This is in a similar form as Equation (2). Equation (18) divides both sides by h, converting the values to frequency equivalents. In Equation (19), α equals the product of 2π times ν_e divided by ν_{re}. Equation (20) rearranges the components and solves for ν_{re}, which equals 2 times π times ν_e divided by α. Equation (21) solves for π, α times ν_{re} divided by 2 times ν_e.

$$E_e = \frac{c^2 \mu_0}{4\pi}\frac{e^2}{r_e} \tag{17}$$

$$\frac{E_e}{h} = \frac{c\mu_0 e^2}{h}\frac{1}{4\pi}\frac{c}{r_e} = \nu_e = \frac{\alpha\nu_{re}}{2\pi} \tag{18}$$

$$\alpha = \frac{2\pi\nu_e}{\nu_{re}} \tag{19}$$

$$\nu_{re} = \frac{2\pi\nu_e}{\alpha} \tag{20}$$

$$\pi = \frac{\alpha\nu_{re}}{2\nu_e} \tag{21}$$

Equations (22) are known relationships of r_e associated with a_0 and the Compton radius of the electron, λ_e. Dividing both sides by c converts them to frequency equivalents. The relationships are related to the square of α as shown in Equation (23), ν_{a0} divided by ν_{re}. There is no 2 in the relationship with ν_{re} as in Equation (24) since this represents a potential energy compared to Equation (10).

$$r_e = \frac{\alpha\lambda_e}{2\pi} = \alpha^2 a_0 \tag{22}$$

$$\frac{r_e}{c} = \alpha^2 \frac{a_0}{c} = \frac{1}{\nu_{re}} = \frac{\alpha^2}{\nu_{a0}} \tag{23}$$

$$\alpha^2 = \frac{\nu_{a0}}{\nu_{re}} = \frac{2\nu_R}{\nu_e} \tag{24}$$

Equation (25) expresses the relationship of the ratio of the smallest and largest hydrogen components of this series from ν_R to ν_{re}. In this case it is related to α^3 divided by 4π. Equation (26) solves for α, and Equation (27) solves for π.

$$\frac{\alpha^3}{4\pi} = \frac{\nu_R}{\nu_{re}} \tag{25}$$

$$\alpha = \sqrt[3]{\frac{4\pi v_R}{v_{re}}} \quad (26)$$

$$\pi = \frac{\alpha^3 v_{re}}{4 v_R} \quad (27)$$

Figure 1 plots these relationships as log values, which demonstrates them more clearly. Many of the possible sums and differences of the log values of 2, 2π, and α exist as ratios of the actual physical components of hydrogen.

Discussion

π, a ubiquitous constant in physics, is frequently incorporated into the definition of constants. In this specific system the π relationships are logical, but not intuitive since they are so convoluted. α can be conceptually viewed as the change in the velocity of the electron from the annihilation speed of c to αc, Equation (8). This concept can also be viewed as a momentum coupling constant, where the square of the velocity of the mass is related to the energy. Therefore the ratio of v_R and v_e is related to α^2, Equation (10). This energy is then related to the ionization energy

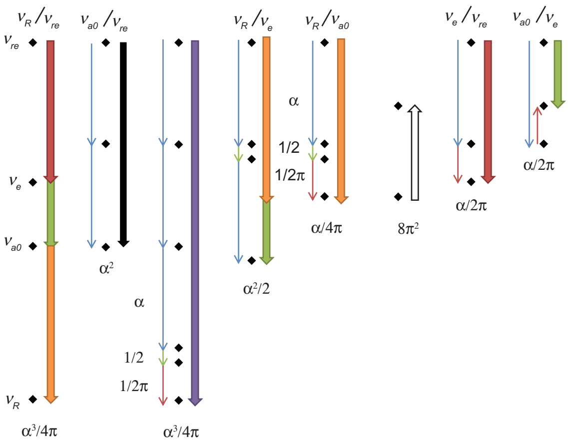

Figure 1. This figure plots the log values for α, thin blue down arrow; 2π, thin red up arrow; $1/2\pi$, thin purple down arrow; and log 1/2, thin green down arrow. The associated log ratio values of the fundamental constants related to the frequency equivalents of v_{re}, v_e, v_{a0}, and v_R are also shown, with their inter- relationships. The thick green down arrow is the of ratio v_{a0}/v_e. The thick red down arrow is the ratio v_e/v_{re}. The thick orange down arrow is v_R/v_e. Most of the possible combinations of these factors exist as physical entities. Note that the series alternates from a distance to a mass/energy entity. There is a ratio change of α, scaled by either 2, 2π, both, or neither. The only value that is independent of α is the $8\pi^2$ ratio, the open up black arrow. The ratio of v_{a0}/v_{re} is the thick solid black arrow and is related to α^2. The purple down arrow is related to v_R/v_{re} and is related to $\alpha^3/4\pi$.

of hydrogen. The factor 2 is related to that fact that this is not a potential energy which is related to the definition of α as a potential energy. The ionization energy is similar to a kinetic energy, in contrast to an annihilation energy.

The 2π factor is related to the transformation of a radius, a_0, to an orbital circumference, which effectively elongates the wavelength equivalent and decreases it energy/frequency equivalent. This change decreases the frequency equivalent value by dividing by 2π into the radius frequency equivalent. This 2π factor is related to the ratio between the a_0 and a circular orbit of the electron, Equation (13). These two factors account for 4π of the $8\pi^2$ ratio change in Equation (15).

The second 2π factor associated with the total $8\pi^2$ ratio arises from the fact that the total ratio relationship of v_R and v_e is defined by Equation (10). Figure 1 demonstrates that any added factor related to the ratio of v_R and v_{a0} must be cancelled out by the ratio relationship of v_{a0} and v_e. The ratio of these two ratios, Equation (15), leads to the square of 2π, Figure 1. The α factors mathematically cancel.

Equation (16) derives π solely from the frequency equivalents of quantum properties of hydrogen and integers. This derivation does not describe the actual geometry of hydrogen in a classic Euclidian geometric sense, but is the result of dividing the ratios between v_{a0} and v_e by v_R and v_{a0}, Figure 1. In this specific and unique situation, three fundamental quantum constant values of identical units are all related to just integers and π, an unusual phenomenon that combines raw quantum physical constants and geometric ones.

r_e is also defined by these same hydrogen factors, and therefore is also related to integers, α, and π. Logically, this is very similar to the relationship between v_R and v_{a0} in Equations (2 and 17). However, r_e is related to a potential energy, which accounts for the difference from the relationship of v_R and v_{a0}.

Figure 1 demonstrates the relative values of these scaling ratios in log format. It is clear that many of the different possible sums of log α, log 2, log 1/2, log 2π, and log $1/2\pi$ are associated with these four fundamental constants and their inter-relationships. α^2 is related to the spread from v_R to v_e, or v_{a0} to v_{re}, and α^3 is related to the maximum ratio transition from v_R to v_{re}. This series alternates between constants related to mass/energy and distance. Each transition is scaled by a factor of α in combination with 2 or 2π. Each α transition can be viewed as a change in velocity with a change in momentum. These transitions are from energy to distance. Distance times a frequency is a velocity. A transition from a mass/energy, v_e, to mass/energy, v_R, or from distance, v_{a0}, to distance, v_{re}, is not associated with a π factor, but is associated with α^2 and integers, since there is no transition from a radius to a circumference.

By looking at electron impedances, Cameron[9] found similar though not identical scaling relationships of α and the properties of hydrogen. The scaling separations were evaluated in a logarithmic form as well, but were less exact than described in this paper. MacGregor's paper[10] is another example where α represents a scaling factor crossing a broad range of physical settings. In this case values near α are the scaling factors between the ratios of the relative lifetimes of many of the metastable elementary particles. For example, α^4 is the scaling between the neutron and muon, the pion^+ to pion^0, and between many of the unpaired mesons and baryons to the paired mesons or baryons. Each different group is also spaced by approximately α. These patterns are quite similar to those described in this work, where a series of entities are spaced by powers of α. Therefore these observations have many potential implications beyond hydrogen in isolation.

What we see is a very complicated, but exact, "dance" between all six of these constants. If only three of the physical constants are known, it is possible to derive all six. It is not typical to consider calculating α from many of these properties, but it is perfectly valid. Equation (5) demonstrates that many other physical constants are directly related to this system as well.

The ratio inter-relationships between all of these factors must also be fulfilled for effective α (running α) as well. In fact, this group of ratios partially explains why there is an effective α at all. The only mathematical means to maintain all of these convoluted ratio relationships, Equations (6, 11, 13, 15, 19, 24 and 26), is for α to change. π is a constant that should not conceptually change, but all of the other factors could. The Bohr radius can be viewed as the minimum "quantum equilibrium" separation of two unit opposite changes in hydrogen. The larger quantum radii possibilities in hydrogen represent the Rydberg series, and α does not change. a_0 can be viewed as the "natural" distance that represents a stable quantum minimum. In hydrogen, a charge distance separation less than the Bohr radius of unit charges is not at a "stable" or a possible classic quantum radius. This unusual setting is secondarily associated with a change in α. In this hydrogen example, the mass of the electron can be viewed as a constant as well. The radius would be less than the Bohr radius, and therefore would have a higher value for the frequency equivalent than v_{a0}. To maintain the $8\pi^2$ relationship of Equation (15), the other factor that must change is the frequency equivalent of the energy. If the inter-relationships were related to the simple ratio values of any two factors alone, then running α would not have to exist. In Equation (15), v_{a0} is squared. In this theoretical representative case, any change of v_{a0} would have to be balanced with the square of that change in v_R. This "forces" the value to change to maintain all of the other scaling relationships. Therefore running α is necessary to fulfill all of these inter-relationships simultaneously and continuously.

Conclusion

Hydrogen is the simplest atom and manifests the most basic inter-relationships of many fundamental physical constants. This paper demonstrates the very complicated, yet logical, inter-relationships of these hydrogen properties. Classic Euclidian geometry and many quantum constants are interwoven. Equation (15) relating π and hydrogen is significant, since it demonstrates that the quantum properties of mass, distance, and energy are all linked to π. The factors π and α demonstrate that there is a deep fundamental geometric and coupling constant relationship that is unified across these different physical entities and their properties. Evaluating them in a common unit allows for this observation, which would otherwise be obscured.

Disclosures

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